A. Giovanidis 16.07.2014

Stochastic Geometry modeling and analysis of wireless networks (part I - theory)

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Outline for hour 1

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What is a Point Process.
- Intro and Fundamentals
- The Poisson Point Process
- Operations preserving the law of Poisson

Conditioning a Point Process.
- Palm Theory
- Palm Measure, Slivnyak's Theorem, and Integration
- Marked Processes

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A.1 - Intro and Fundamentals

Point Process in 1D (1)

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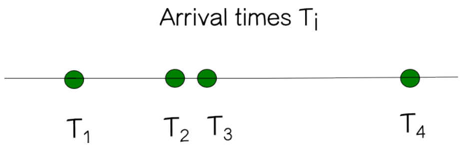
- A point process in one dimension ("time") is a useful model for the sequence of random times when a particular event occurs.
- Example "emergency calls in a hospital". Each call happens at an instant (point) of time.
- Given a period of time there will be a random number of such calls.



Point Process in 1D (2)

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- 1D Point Processes (PP) has a natural ordering.
- Arrival times: $T_1 < T_2 < \ldots$ (dependent)
- Inter-arrival times: $S_i = T_{i+1} - T_i$ (independent for some)
- Cumulative counting process: $N_t = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \leq t\}}$
- Interval counts: $N(a, b] = N_b - N_a$

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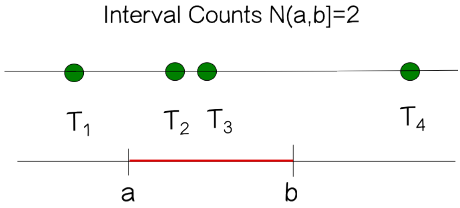
- Interval counts: $N(a, b] = N_b - N_a$

$N_t =$ total number of points up to time t, for all $t \geq 0$



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Point Process in $\geq$ 2D (1)

- In higher dimensions there is no natural ordering of points.
- No natural analogue for inter-arrival times S_i, counting process N_t.
- BUT! Generalise the interval counts to region counts.
- Superposition of PPs: $N(A) = N_1(A) + N_2(A)$ (additive).

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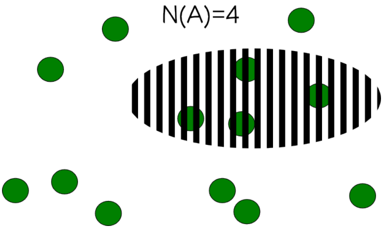
$$N(A) = number\ of\ points\ falling\ in\ A, \quad A \subset \mathbb{R}^d$$



Point Process in $\geq$ 2D (2)

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- It is interesting to study a PP using only the vacancy indicators.

$$\begin{aligned} V(A) &= \mathbf{1}\{N(A) = 0\} \\ &= \mathbf{1}\{\textit{there are no points falling in } A\}. \end{aligned}$$

- Superposition of PPs: $V(A) = V_1(A) \cdot V_2(A)$ (multiplicative).

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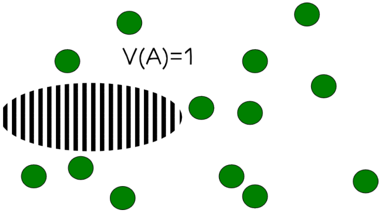

Point Process Fundamentals (1)

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- A spatial point process Φ is a random, finite or countably-infinite collection of points in the space $\mathbb{R}^d$ without accumulation points.
- Realization: Discrete subset $\phi = \{x_i\} \subset \mathbb{R}^d$. The indices of points follow some random enumeration.
- Counting measure: $N(A)$ gives the number of points of ϕ in A. $\mathbf{1}_{\{x \in A\}}$ is the Dirac measure at x.

$$N(A) = \sum_i \mathbf{1}_{\{x_i \in A\}}$$

Point Process Fundamentals (2)

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- For all real functions f defined on $\mathbb{R}^d$ we have

$$\sum_i f(x_i) = \int_{\mathbb{R}^d} f(x)\, N(dx).$$

- No accumulation points $\Rightarrow N(A) < \infty$ for any bounded A.
- Simple PP $\Rightarrow \mathbb{P}[N(\{x\}) \leq 1,\ \forall x] = 1$, i.e. the points $\{x_i\}$ are pairwise different a.s..
- The distribution of a PP Φ is entirely characterised by the family of finite distributions $\{N(A_1), \ldots, N(A_k)\}$, with $A_1, \ldots, A_k$ bounded.

Characteristics of the Point Process

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- The intensity (mean) measure

$$\Lambda(A) \;=\; \mathbb{E}[N(A)].$$

- The empty (void) probability

$$\mathcal{V}(A) \;=\; \mathbb{P}[N(A)=0],\; A \in \mathcal{B}.$$

- The Laplace functional, for any function $f : \mathbb{R}^d \to \mathbb{R}_+$

$$\mathcal{L}(f) \;=\; \mathbb{E}\left[e^{-\int_{\mathbb{R}^d} f(x)N(dx)}\right].$$

It completely characterises the distribution of the PP. For $f(x) = \sum_{i=1}^{k} t_i \mathbf{1}_{(x\in A_i)} \Rightarrow$ $\mathcal{L}(f) = \mathbb{E}\left[e^{-\sum_i t_i N(A_i)}\right]$ is the joint LT of the random vector $\{N(A_1), \ldots, N(A_k)\}$.

Campbell's Formula

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- Let f be a measurable function, then the equality holds

$$\mathbb{E}\left[\int_{\mathbb{R}^d} f(x)\, N(dx)\right] = \int_{\mathbb{R}^d} f(x)\, \Lambda(dx)\,.$$
$$\stackrel{(int.\ fun.)}{=} \int_{\mathbb{R}^d} f(x)\, \beta(x)\, dx.$$

If the intensity measure Λ satisfies $\Lambda(A) = \int_A \beta(x)\, dx$ for some function β, then the latter is the intensity function of Φ.

$$\mathbb{P}[N(dx) > 0] \approx \mathbb{E}[N(dx)] \approx \beta(x)\, dx.$$

Stationarity and Isotropy

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A PP Φ is called:

1. **Stationary** (homogeneous) if the distribution of Φ is invariant with respect to translation of the origin, i.e.

$$\{X_i\} \stackrel{d.}{=} \{X_i - u\}, \ \forall u \in \mathbb{R}^d.$$

2. **Isotropic** if stationary & the distribution of Φ is invariant with respect to rotation of the origin.

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A.2 - The Poisson Point Process

The Poisson Point Process

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Definition:

The Poisson PP Φ of intensity measure Λ is defined by means of its finite-dimensional distributions:

$$\mathbb{P}\left\{N(A_1) = n_1, \ldots, N(A_k) = n_k\right\} \quad = \quad \prod_{i=1}^{k} \left(e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^{n_i}}{n_i!} \right),$$

for every $k = 1, 2, \ldots$ and all bounded, mutually disjoint sets A_i, for $i = 1, \ldots, k$.

If $\Lambda(dx) = \lambda dx \Rightarrow$ homogeneous.

Properties

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- For every compact set $A \subset \mathbb{R}^d$ the count $N(A)$ has a Poisson distribution with mean $\Lambda(A)$. Homogeneous case: $\lambda \mathcal{S}(A)$.

- Complete Independence Property
  If $A_1, \ldots, A_k$ are mutually disjoint compact sets, then $N(A_1), \ldots, N(A_k)$ are independent random variables.

- Laplace Functional

$$\mathcal{L}(f) = e^{-\int_{\mathbb{R}^d}\left(1-e^{-f(x)}\right)\Lambda(dx)}.$$

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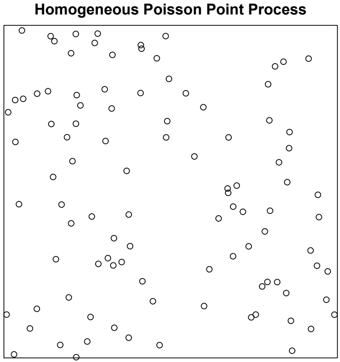
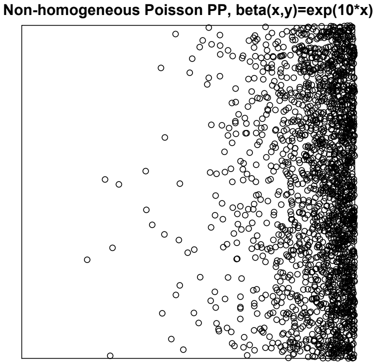
Examples





Simulating the Poisson PP (1)

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Conditional Property

Consider a Poisson PP in $\mathbb{R}^d$ with uniform intensity $\beta(x) = \lambda > 0$. Let $W \subset \mathbb{R}^d$ be any bounded region with $0 < \mathcal{S}(W) < \infty$.
Given $N(W) = n$, the conditional distribution of $N(B)$ for $B \subseteq W$ is binomial:

$$\mathbb{P}\left[N(B) = k | N(W) = n\right] = \binom{n}{k} p^k (1-p)^{n-k},$$

where $p = \frac{\mathcal{S}(B)}{\mathcal{S}(W)}$.

- Given there are n points of the Poisson PP in W, these are conditionally independent and uniformly distributed in W.

Simulating the Poisson PP (2)

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How to simulate a Poisson PP of intensity λ in W?

1. First generate a random variable with Poisson distribution and mean $\lambda \mathcal{S}(W)$.
2. Given the realisation of the Poisson r.v. n, generate n independent uniform random point in W.

Simulating the Poisson PP (3)

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To see why this procedure produces a Poisson PP let us use the Laplace functional

$$\begin{aligned}
\mathcal{L}(f) &= \mathbb{E}\left[\mathbf{1}_{\{N=0\}} + \mathbf{1}_{\{N>0\}} e^{-\sum_{k=1}^{N} f(X_i)}\right] \\
&= e^{-\Lambda(W)} \sum_{k=0}^{\infty} \frac{\Lambda(W)^k}{k!} \left(\int_W e^{-f(x)} \frac{\Lambda(dx)}{\Lambda(W)}\right)^k \\
&= e^{-\Lambda(W) + \int_W e^{-f(x)} \Lambda(dx)} \\
&= e^{-\int_W \left(1 - e^{-f(x)}\right) \Lambda(dx)}.
\end{aligned}$$

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A.3 - Operations preserving the law of Poisson

Operation I: Mapping

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Consider a fixed transformation $g : \mathbb{R}^d \to \mathbb{R}^{d'}$ to each point x_i of Φ.
Let $g^{-1}(A)$ be bounded for each $A \in \mathbb{R}^{d'}$ bounded.

- The resulting process has measure $N_M(A) = \sum_i \mathbf{1}_{\{g(x_i) \in A\}}$.
- For a Poisson PP with intensity Λ on $\mathbb{R}^d$, its transformation by g is also a Poisson PP with intensity $\Lambda'(\cdot) = \Lambda\left(g^{-1}(\cdot)\right)$.

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Examples

Dilation: $g(x) = \gamma x$, $\gamma > 0$, $\Lambda'(A) = \Lambda(A/\gamma)$.

Vector Translation: $g(x) = x + v$, $v \in \mathbb{R}^d$, $\Lambda'(A) = \Lambda(A - v) \overset{u.}{=} \lambda \mathcal{S}(A)$.

Polar coordinates: $g(x) = (|x|, \angle x)$, $\Lambda'([0,r), [0,\theta)) = \lambda \pi r^2 \theta / (2\pi)$.

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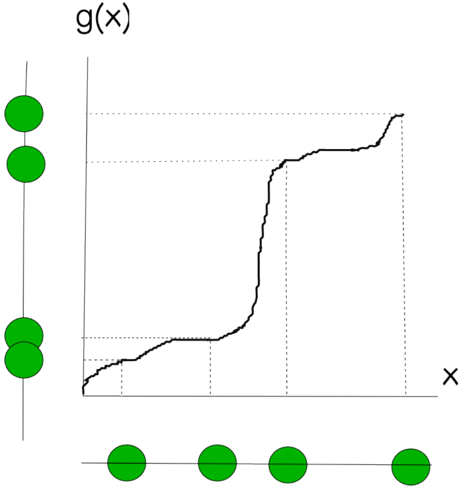
Mapping Example



Operation II: Thinning

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In thinning some fraction of points of the PP Φ are deleted.

- The thinning is a PP Φ_T given by $N_T(A) = \sum_i \Delta_i \mathbf{1}_{\{x_i \in A\}}$. The indicators Δ_i define which atom will stay.

Independent Thinning

The r.v.'s Δ_i are independent and $\mathbb{P}[\Delta_i = 1] = \delta(x_i)$.

The independent thinning of a Poisson PP of intensity measure Λ with the retention probability δ yields a Poisson PP of intensity measure $\delta\Lambda$.

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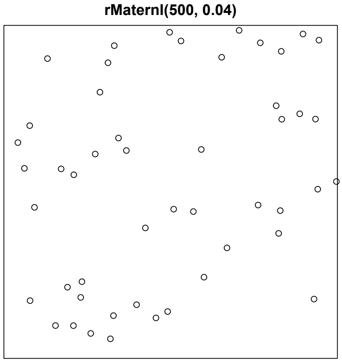
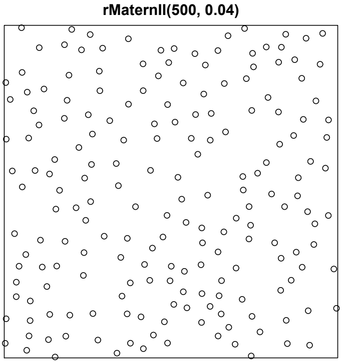
Dependent Thinning - Matérn Model I & II





Operation III: Superposition

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- The superposition of PPs Φ_k is defined as the sum $\Phi = \sum_k \Phi_k$.
- Viewed either as the union of sets or as the union of measures.
- If $N_{\Phi_1}(A)$ and $N_{\Phi_2}(A)$ are the number of points in region $A \subseteq \mathbb{R}^d$, then $N_{\Phi_1+\Phi_2}(A) = N_{\Phi_1}(A) + N_{\Phi_2}(A)$.

The superposition of independent Poisson PPs with intensities Λ_k is a Poisson PP with intensity measure $\sum_k \Lambda_k$ if and only if the latter is a locally finite measure.

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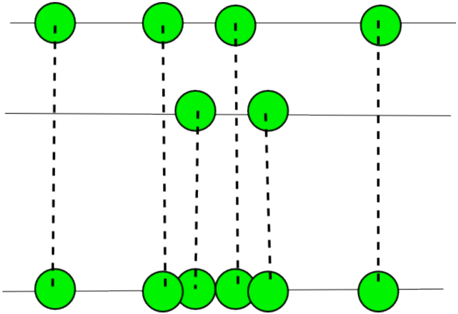
Superposition



Operation IV: Cluster Formation

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- Starting by a process Φ replace each point $x_i \in \Phi$ by a random finite set of points Z_i (the cluster).
- The superposition of all clusters yields $\Phi_C = \bigcup_i Z_i$.

Example: Matérn cluster process where the parent process is uniform Poisson and each cluster Z_i consists of a random Poisson number of points with mean μ, uniformly distributed in the disc $\mathcal{B}(x_i, \mu)$.

Matérn Cluster

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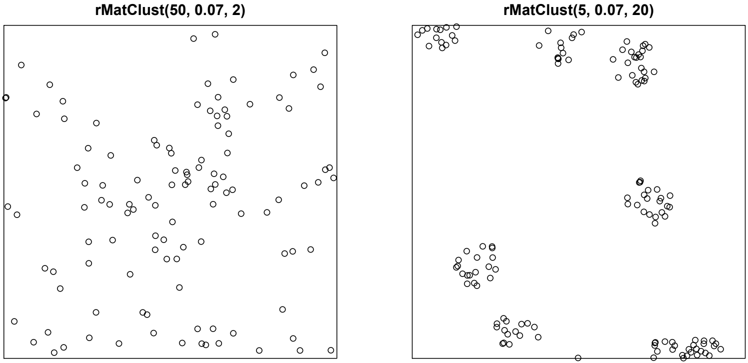

Outline for hour 2

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What is a Point Process.
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- Palm Measure, Slivnyak's Theorem, and Integration
- Marked Processes

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B.1 - Palm Theory

Motivation (1)

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- Interested in properties relating to a typical point of the process.
- Conditional probabilities of events **given there is a point of the PP at a specific location**.
- Concept of Palm distribution and Campbell-Mecke formula.

Motivation (1)

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- Interested in properties relating to a typical point of the process.
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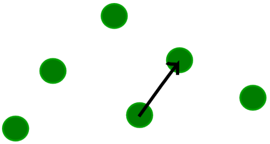
- Concept of Palm distribution and Campbell-Mecke formula.

Q: What is the probability distribution of
- the distance from some atom of Φ to its nearest neighbour atom?



Nearest Neighbour Distance

Motivation (2)

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- Formally $R_x = dist\left(x, \Phi \setminus x\right)$, $x \in \Phi$. Find:

$$\mathbb{P}\left(R_x \leq r | x \in \Phi\right).$$

- Non-elementary conditioning: The event $\{x \in \Phi\}$ has 0 probability.

For the case of stationary PPs e.g. uniform Poisson

Condition on the event that there is a point of Φ at the origin 0.

Motivation (3)

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- Let Φ be Poisson PP with intensity λ.
- Take a small neighbourhood around 0, e.g. the ball $\mathcal{B}(0, \epsilon)$.
- $R_\epsilon = dist(0, \Phi \setminus \mathcal{B})$
  the distance from 0 to the nearest point of Φ outside $\mathcal{B}$.

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Motivation (3)

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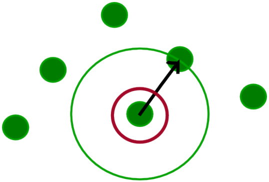
- Take a small neighbourhood around 0, e.g. the ball $\mathcal{B}(0, \epsilon)$.
- $R_\epsilon = dist(0, \Phi \setminus \mathcal{B})$
  the distance from 0 to the nearest point of Φ outside $\mathcal{B}$.

$$\begin{aligned} \mathbb{P}[R_\epsilon > r | N(\mathcal{B}(0,\epsilon)) > 0] &= \exp\left\{-\lambda\pi\left(r^2 - \epsilon^2\right)\right\} \\ \mathbb{P}[R_0 > r | 0 \in \Phi] &= \exp\left\{-\lambda\pi r^2\right\}. \end{aligned}$$



Nearest Neighbour Distance

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B.2 - Palm Measure, Slivnyak's Theorem, and Integration

Palm Distribution

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C. Palm (1907-1951) for the study of telephone traffic.

- The **Palm probability** $\mathbb{P}^x(\Gamma)$ of an event Γ at location x is the conditional probability that the event Γ will occur, given $x \in \Phi$.

- If the events are restricted to the outcome events (counts of atoms) of a PP, we have the **Palm distribution** $\mathbf{P}^x(\Gamma)$ for the event Γ of a PP, defined as

$$\mathbf{P}^x(\Gamma) = \mathbb{P}^x(\Phi \in \Gamma).$$

Campbell Measure

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- Reminder: Mean Radon measure

$$\Lambda(A) = \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbf{1}_{\{x \in A\}} N(dx)\right] = \mathbb{E}[N(A)].$$

- We define the Campbell measure

$$C(A \times \Gamma) = \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbf{1}_{\{x \in A\}} \mathbf{1}_{\{\Phi \in \Gamma\}} N(dx)\right] = \mathbb{E}\left[N(A)\, \mathbf{1}_{\{\Phi \in \Gamma\}}\right].$$

Palm Probability Measure

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Radon-Nikodým Theorem expresses the two measures in relation as

$$C\left(A\times\Gamma\right) \;=\; \int_A \mathbf{P}^x\left(\Gamma\right)\Lambda\left(dx\right).$$

$\mathbf{P}^{x}\cdot\left(\Gamma\right)=\frac{dC(\cdot\times\Gamma)}{d\Lambda(\cdot)}$ is the R-N derivative. It is also a probability measure.

Reduced Palm Measure

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Let us remove the extra point x from the consideration. The reduced Palm distribution $\mathbf{P}^{!x}$ of a PP Φ is the distribution of $\Phi \setminus x$ under $\mathbf{P}^x$:

$$\mathbf{P}^{!x}(\Gamma) = \mathbf{P}^x(\Phi \setminus \{x\} \in \Gamma).$$

The relevant reduced Campbell measure of Φ is

$$\begin{aligned} C^!(A \times \Gamma) &= \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbf{1}_{\{x \in A\}} \mathbf{1}_{\{\Phi \setminus x \in \Gamma\}} N(dx)\right] \\ &= \int_A \mathbf{P}^{!x}(\Gamma)\Lambda(dx) \end{aligned}$$

Slivnyak-Mecke Theorem (1)

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The Poisson PP, under the Palm distribution $\mathbf{P}^x$, behaves as if it were a Poisson PP superimposed with a fixed point at the location x

$$\begin{aligned} \mathbf{P}^x &= \mathbf{P} * \Delta_x \quad (\textit{convolution}) \\ \Phi^x &\stackrel{d.}{=} \Phi \cup \{x\}. \end{aligned}$$

Slivnyak-Mecke Theorem (2)

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Theorem S-M: Let Φ be a PP with locally finite intensity measure Λ. Suppose the distributions $\mathbf{P}$ and $\mathbf{P}^x$ of Φ are related by

$$\mathbf{P}^x = \mathbf{P} * \Delta_x.$$

Then Φ is a Poisson PP with intensity measure Λ.

Theorem

Φ is a Poisson PP **if and only if**

$$\mathbf{P}^{!x} = \mathbf{P}.$$

Palm Distribution and Stationarity

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Palm distributions $\mathbf{P}^x$ are equivalent under translation:

If Φ is a stationary point process in $\mathbb{R}^d$ (e.g. Poisson), then

$$\Phi^x \stackrel{d.}{=} \Phi^0 + x,$$

where Φ^x again denotes a process governed by the Palm probability measure $\mathbb{P}^x$. More formally,

$$\mathbf{P}^x = T_x \mathbf{P}^0,$$

where T_x is the effect of translation by a vector x, $T_x N(A) = N(A - x)$.

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Application: Nearest Neighbour Function

For a stationary PP Φ in $\mathbb{R}^d$, the nearest neighbour function G is the distribution of the distance from a typical point $x \in \Phi$ to the nearest other point of Φ

$$G(r) \;=\; \mathbb{P}^x\left(dist\left(x, \Phi \setminus \{x\}\right) \leq r\right) = \mathbf{P}^x\left(N(\mathcal{B}\left(x, r\right) \setminus \{x\}) > 0\right).$$

- Stationarity $\Rightarrow$ the function does not depend on x.

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Application: Nearest Neighbour Function

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- Stationarity $\Rightarrow$ the function does not depend on x.
- If Poisson PP we may use *Slivnyak's* theorem $\mathbf{P}^{!x} = \mathbf{P}$

$$\begin{aligned} G(r) &= \mathbb{P}^0 \left(dist \left(0, \Phi \setminus \{0\} \right) \leq r \right) = \mathbf{P} \left(dist \left(0, \Phi \right) \leq r \right). \\ &= 1 - \exp\left(-\lambda \pi r^2 \right), \end{aligned}$$

equal to the empty space function.

Campbell-Mecke Formula

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We want to calculate sums over Φ for functions that depend on the atom in x **and** on the entire PP.

For any $f(x, \Phi)$ that is integrable with respect to the Campbell measure,

$$\mathbb{E}\left[\sum_i f(x_i, \Phi)\right] = \int_{\mathbb{R}^d} \mathbb{E}^x\left[f(x, \Phi)\right] \Lambda(dx).$$

Campbell-Mecke Formula

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For Poisson PP, use the *reduced Palm measure* and *Slivnyak's theorem*

$$\mathbb{E}\left[\sum_i f(x_i, \Phi \setminus \{x_i\})\right] = \int_{\mathbb{R}^d} \mathbb{E}\left[f(x, \Phi)\right] \Lambda(dx).$$

C-M Application: Dependent Thinning

Consider the Matern I PP: Results from thinning a Poisson PP

- Original Poisson PP Φ of intensity λ in $\mathbb{R}^2$.
- Delete any $x \in \Phi$: $dist\left(x, \Phi \setminus \{x\}\right) \leq r_0$ (any point with neighbour in Φ closer than r_0). Resulting PP Ψ.
- $f\left(x, \Phi\right) = \mathbf{1}_{\{x \in A\}} \mathbf{1}_{\{dist(x, \Phi \setminus \{x\}) > r_0\}}$.
- $\mathbb{P}^x\left(dist\left(x, \Phi \setminus \{x\}\right) > r_0\right) = \exp\left(-\lambda \pi r_0^2\right)$.

C-M Application: Dependent Thinning

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- $f\left(x, \Phi\right) = \mathbf{1}_{\{x \in A\}} \mathbf{1}_{\{dist(x, \Phi \setminus \{x\}) > r_0\}}$.
- $\mathbb{P}^x\left(dist\left(x, \Phi \setminus \{x\}\right) > r_0\right) = \exp\left(-\lambda \pi r_0^2\right)$.

$$\begin{aligned} \mathbb{E}\left[N\left(\Psi \cap A\right)\right] &= \mathbb{E}\left[\sum_i f\left(x_i, \Phi\right)\right] = \int_{\mathbb{R}^d} \mathbb{E}^x\left[f\left(x, \Phi\right)\right] \lambda dx = \\ &= \lambda \mathcal{S}\left(A\right) \exp\left\{-\lambda \pi r_0^2\right\} \Rightarrow \lambda_\Psi = \lambda \exp\left\{-\lambda \pi r_0^2\right\}. \end{aligned}$$

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B.3 - Marked Processes

Marked PP (1)

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- The points of a PP might be labelled with extra information called marks.
- e.g. If emergency calls are a PP on the line, each point might carry a label with the place of the call and the nature of emergency.
- Marked point is a pair: $\{x_i, m_i\}$, $m_i \in M \subseteq \mathbb{R}^k$.
- The marked PP is denoted by $\tilde{\Phi}$ and has counting measure $\tilde{N}(\cdot)$, i.e.

$$\tilde{N}(A \times M) = \sum_i \mathbf{1}_{\{x_i \in A, m_i \in M\}}.$$

Marked PP (2)

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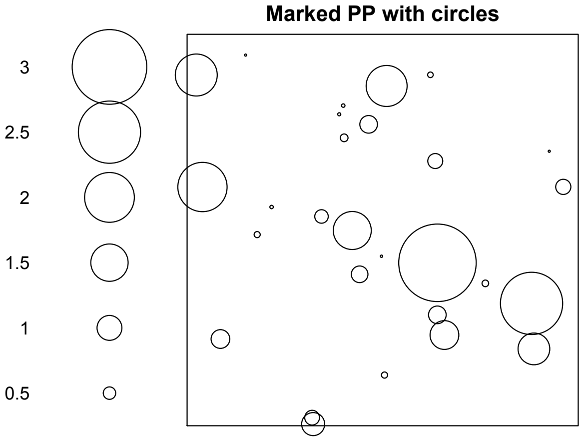

Marked PP (3)

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- The corresponding projected PP of $\tilde{\Phi}$ on $\mathbb{R}^d$ should be locally finite

$$\tilde{N}(A \times M) < \infty.$$

- A marked PP constructed by adding independent random marks to a Poisson PP is equivalent to a Poisson PP in the product space.

- Thinning a PP is formalised by a marked point process with marks in $\{0, 1\}$. $\{0\}$=retain and $\{1\}$=delete.

Marked Intensity

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- The intensity measure of $\tilde{\Phi}$ is a measure on $A \times M$ determined by

$$\tilde{\Lambda}(A \times M) = \mathbb{E}\left[\tilde{N}(A \times M)\right] = \mathbb{E}\left[\sum_{(x,m)\in\tilde{\Phi}} \mathbf{1}_{\{x\in A\}}\mathbf{1}_{\{m\in M\}}\right].$$

- Campbell's formula takes the form

$$\mathbb{E}\left[\sum_{(x,m)\in\tilde{\Phi}} f(x,m)\right] = \int_{A\times M} f(x,m)\,\tilde{\Lambda}(dx,dm).$$

Independent Marking (1)

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- Independently Marked PP (i.m.PP): If given the locations of the points $\Phi = \{x_i\}$, the marks are mutually independent random vectors in M and if the conditional distribution satisfies

$$\mathbf{P}\left(m \in \cdot|\Phi\right) = \mathbf{P}\left(m \in \cdot|x\right) = Q_x\left(dm\right).$$

- The Laplace functional of an i.m.PP for all functions in $\mathbb{R}^+$ is

$$\begin{aligned} \mathcal{L}_{\tilde{\Phi}}\left(f\right) &= \mathbb{E}\left[e^{-\sum_i f(x_i,m_i)}\right] \\ &= e^{-\int_{\mathbb{R}^d}\left(1-\int_M e^{-f(x,m)}Q_x(dm)\right)\Lambda(dx)}. \end{aligned}$$

Independent Marking (2)

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The reduced Campbell-Mecke formula for i.m.PP gives

$$\mathbb{E}\left[\sum_i f\left(x_i, m_i, \Phi \setminus \{x_i\}\right)\right] = \int_{\mathbb{R}^d}\int_M \mathbb{E}^{!x}\left[f\left(x, m, \tilde{\Phi}\right)\right] Q_x(dm)\Lambda(dx),$$

and for the Poisson PP case

$$\mathbb{E}\left[\sum_i f\left(x_i, m_i, \Phi \setminus \{x_i\}\right)\right] = \int_{\mathbb{R}^d}\int_M \mathbb{E}\left[f\left(x, m, \tilde{\Phi}\right)\right] Q_x(dm)\Lambda(dx).$$

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Marked Stationarity

- A marked point process on $\mathbb{R}^d$ with marks in M is stationary if its distribution is invariant under shifts of $\mathbb{R}^d$ only, $\forall v \in \mathbb{R}^d$

$$(x, m) \quad \rightarrow \quad (x + v, m).$$

- The location of the point x is shifted but the mark stays unchanged.

- For a stationary marked PP, the intensity measure takes the form

$$\mathbb{E}\left[\tilde{N}(A \times M)\right] \quad = \quad \lambda \mathcal{S}(A)\, Q_0(M).$$

$\lambda =$ expected number of points per unit area, $Q_0(M) =$ distribution of the typical mark (probability measure).

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END OF PART I