An Optimal Stopping Approach to ARQ Protocols with Variable Success Probabilities per Retransmission

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***Abstract*— In the current work the conceptual framework of optimally stopping a stochastic process is used to determine the optimal maximum number of retransmissions in an ARQ chain. The process sequentially observed is the binary ARQ feedback after each packet (re)transmission (ACK/NAK). A reward-cost process Y_n^C is constructed as a function of the observed sequence up to time n with a certain reward and cost per trial as well as a final penalty in case the retransmission process is finalised before correct packet reception. Two problems are investigated, namely the cases without and with cost. In the ARQ stopping problem without cost ergodicity conditions of the ARQ Markov chain are stated and proved. These guarantee with probability one finite waiting times until the first ACK is received. The solution of the ARQ stopping problem with cost provides an explicit expression for the optimal truncation time of ARQ protocols as a function of the costs and rewards and suggests a tradeoff between delay and dropping probability. Conditions for cases when the ARQ chain should not be truncated as well as when no retransmissions should be allowed at all are presented. The stopping rule is applied to practical ARQ scenarios where the behavior of the truncation time with respect to different supported rate, delay and dropping is investigated.**

I. Introduction

In wireless communication networks, the stochastic nature of the channel provides an unreliable link to nodes that attempt to communicate with each other. Noise and channel fading set capacity limits on the instantaneous rate of information that can be transmitted through the link, error free. However, 100% reliable communications can not be fulfilled at any rate in practice where only finite-length codes and imperfect channel state information are available. Thus communications is always bound to errors, which can be diminished with the aid of small size modulation constellations, low-rate error-control codes and expensive channel state information feedback.

An alternative approach to deal with the erroneous behavior of the channel is to rely on an Automatic Retransmission Request (ARQ) protocol which repeats transmission of packets declared in error at the receiver. In this case the transmitter is informed through a control channel and binary feedback (ACK/NAK) whether it should retransmit the erroneous message or move on to the first transmission of the next packet waiting for service. On the one hand since retransmissions are only activated when necessary, system throughput can be improved relative to the use of Forward Error Correction Codes (FEC). Combination of these two techniques to combat channel errors has given rise to Hybrid ARQ (HARQ) protocols [1], [2]. On the other hand, occasionally even with HARQ, a large number of retransmissions may be required resulting in an unacceptable maximum delay. This delay can be reduced by limiting the maximum allowable retransmission number leading to truncated ARQ techniques [3], at the expense of packet loss when the maximum number is exceeded. A cross-layer combination of adaptive modulation and coding with truncated ARQ has been investigated in [4] where given a maximum number of retransmissions and a maximum acceptable probability of packet loss that satisfy certain service quality requirements, the gain in spectral efficiency is shown to be considerable. Furthermore this improvement decreases as the allowed retransmissions per packet increase.

In most current approaches in the literature that deal with ARQ, the truncated version is generally accepted as realistic and optimal in terms of delay-throughput tradeoff. However the maximum number of retransmissions is considered as a predefined constant, given which the entire analysis follows, see e.g. [2]–[5], [6]. In the current work we make use of the conceptual framework of sequential analysis and optimal stopping [7] to determine the optimal accepted number of retransmissions in an ARQ chain, given a sequence of rewards and costs per retransmission and a terminal cost when the packet fails to be correctly received. The costs per trial as well as final cost are related to the desired quality of service.

In the following we formulate the ARQ problem as an Optimal Stopping problem, in section II, considering an infinite horizon. The stochastic process sequentially observed is the binary feedback after each packet (re)transmission

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$\{X_n\}$. We construct a reward-cost process $\left\{Y_n^C\right\}$ as a function of the observed sequence up to time n, which will be denoted as the *payoff*. The reward sequence $\{C_n\}$ can be related to some rate gain for successful transmission, whereas the costs can be interpreted as a power/delay cost per retransmission $\{D_n\}$ as well as a final cost in case of dropping at step n, equal to $\{\mu_n\}$.

After each observation of X_n we can decide whether we want to stop and receive the related instantaneous payoff Y_n^C or allow a new retransmission. We are looking for a stopping rule T to maximize the expected payoff providing us with the optimal truncation time. In section III, the ARQ problem with reward and no cost is considered where the retransmissions are not penalized. It is shown that it is obviously optimal to continue retransmissions until the first ACK is received and immediately stop afterwards. Several conditions for finite waiting time (finite trials) up to first ACK received are provided. The maximum expected reward over all possible stopping rules for the problem without cost serves as an upper bound for the case with cost to be presented in section IV.

In the ARQ problem with cost we characterize the maximum expected payoff and find an explicit optimal stopping rule to achieve this, as a function of the stepwise error probabilities, costs per trial and final cost. Given the optimal stopping solution, section V discusses the case of truncation for T=1, where no retransmissions are allowed and we have to live with the erroneous channel since ARQ is not 'worthy' enough in terms of delay or power cost. We consider furthermore conditions when the ARQ chain should not be truncated $T=\infty$. In section VI the rules are applied to scenarios with specific choice of costs $\{C_n\}$ and rewards $\{D_n\}$, $\{\mu_n\}$ related to real ARQ systems and plots illustrate the relative behavior of the optimal retransmission number. Generally speaking the truncation time T is shown to be increasing w.r.t to supporting rate R and dropping cost μ and decreasing w.r.t. delay and/or power cost per retransmission D. Finally section VII draws the conclusions of our work.

II. ARQ as an Optimal Stopping Problem

A. On Optimal Stopping Rules

Let us consider a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbf{P})$ where $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability triple and $\{\mathcal{F}_n : n \in \mathbf{N}\}$ is a filtration, that is an increasing family of sub-σ-algebras of $\mathcal{F}$: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}$. Each $\mathcal{F}_n$ contains all the null sets of $\mathcal{F}$. We consider further a stochastic process $X = (X_n : n \geq 0)$ defined on this probability space each random variable X_n having state space $\mathcal{R}$, measurable with respect to the Borel σ-algebra $\mathcal{B}(\mathcal{R})$. The process is called *adapted* to the filtration $\{\mathcal{F}_n\}$, meaning that for each n, X_n is $\mathcal{F}_n$-measurable. To simplify we consider the case of the natural filtration where $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. Since the process is adapted the value $X_n(\omega)$, $\omega \in \Omega$ is known to us at time n.

The problem of optimal stopping can be described as follows. We observe the sequence of random variables $\{X_1, \ldots, X_n, \ldots\}$ until we decide at some step n to stop and receive a payoff $Y_n(\omega) = f_n(X_1(\omega), \ldots, X_n(\omega))$, which is an $\mathcal{F}_n$ measurable function $f_n : \Omega \to \mathbf{R}$, $f_n^{-1} : \mathcal{B}(\mathcal{R}) \to \mathcal{F}_n$. A random variable $\tau = \tau(\omega) : \ \Omega \to \{1, 2, \ldots, \infty\}$ defined in $(\Omega, \mathcal{F}, \mathbf{P})$ is a *Stopping Time* if it is almost surely (a.s.) *finite*

$$\mathbf{P}\{\tau(\omega) < \infty\} = 1 \quad (1)$$

and satisfies the *non-anticipativity* requirement [8], that is for each $n \in \mathbf{N} \cup \{\infty\}$

$$\{\omega : \tau(\omega) \leq n\} \in \mathcal{F}_n \quad (2)$$

where $\mathcal{F}_\infty := \sigma\left(\bigcup_n \mathcal{F}_n\right) \subseteq \mathcal{F}$. In simple words, the stopping time is a time when we decide to stop our process based solely on the already available samples that we have observed up to and including time n $\{X_1(\omega), \ldots, X_n(\omega)\}$.

We are looking for a stopping rule $\tau = T$ with the attributes (1) and (2) that maximizes the expected reward $\mathbf{E}[Y_\tau]$ in the class of all stopping times $\mathcal{C}$ for which the expectation exists.

Writing $Y = Y^+ - Y^-$, where $Y^+ = \max\{0, Y\}$ and $Y^- = \max\{-Y, 0\}$, the expectation is defined if one of the two terms is finite [9]. Furthermore $Y_\tau \leq \sup_n Y_n$. Hence under the condition that

$$\mathbf{E}\left[\sup_n Y_n^+\right] < \infty \quad (3)$$

the expectation is always well defined, possibly infinite and it holds in particular $-\infty \leq \mathbf{E}[Y_\tau] \leq \mathbf{E}[\sup_n Y_n^+] < \infty$.

The maximum expected reward equals

$$V := \sup_{\tau \in \mathcal{C}, 1 \leq \tau < \infty} \mathbf{E}[Y_\tau] \quad (4)$$

and we are looking for the rule $T \in \mathcal{C}$ (if it exists) such that

$$\mathbf{E}[Y_T] = V \quad (5)$$

Under assumption (3) and if $\mathbf{P}(T < \infty) = 1$, such an optimal stopping rule can be shown to exist [10], [11]).

The rule maximizing the expected return is given by the *principle of optimality* [7], [10] which is the basis of Dynamic Programming.

It suggests that we ought to continue the observations as long as the future expected payoff is greater than the present reward and stop immediately otherwise.

B. The ARQ Model

In the case of ARQ the evolving process is the feedback to the transmitter which contains the information whether a message has been correctly or erroneously received. The observed discrete-time process with finite state space $\{0, 1\}$ is described as follows

$$X_n = \begin{cases} 1 & \text{if NAK is received} \\ 0 & \text{if ACK is received} \end{cases} \quad (6)$$

Note here that usually the value 1 represents a successful reception instead of unsuccessful, however this convention is adopted to simplify notation in the following.

We further define the process $\{Z_n\}$ with countably infinite state space $\{1, 2, ...\}$, where the random variable Z_n is the number of retransmission effort at the n-th time slot. Suppose that at slot $n-1$ we are at the k-th trial of some message, in other words $Z_{n-1} = k$. There are two possibilities for the states of variable Z_n, either to move on to stage $Z_n = k+1$ if a negative acknowledgement comes or to return to stage $Z_n = 1$ for the first transmission of the next message in case of ACK. The events $\{Z_n = k | Z_{n-1} = k-1\}$ and $\{Z_n = 1 | Z_{n-1} = k-1\}$ are mutually exclusive and exhaustive and occur with probabilities p_n and q_n respectively which sum up to 1. Since the process at time slot n depends only on the state of the process at the previous slot, it forms a Markov Chain. The transition probability diagram is given in the following figure and illustrates a specific type of random walk in one-dimension better known as a *success run* while the one step transition

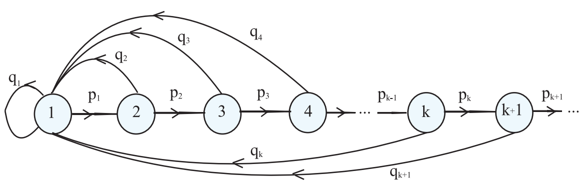

Fig. 1. Transition Probability Diagram for homogeneous ARQ with countably infinite states

probability matrix for the process $\{Z_n\}$ is

$$\mathbf{P} = \begin{pmatrix} q_1 & p_1 & 0 & 0 & 0 & \cdots \\ q_2 & 0 & p_2 & 0 & 0 & \cdots \\ q_3 & 0 & 0 & p_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \tag{7}$$

We assume in our model different success probabilities per trial since the coding, modulation or power per retransmission may be chosen to vary [4], [12] in an effort to minimize the dropping probability and the average number of efforts until ACK. For the process $\{X_n\}$ we have then that $\mathbf{P}(X_n = 1 | Z_n = k) = \mathbf{P}(Z_{n+1} = k+1 | Z_n = k) = p_k$ and $\mathbf{P}(X_n = 0 | Z_n = k) = \mathbf{P}(Z_{n+1} = 1 | Z_n = k) = q_k$. That is, the current value of success probability depends on the number of consecutive unsuccessful retransmissions up to this point. The expected value $\mathbf{E}[X_n] = 0 \cdot q_k + 1 \cdot p_k \leq 1$.

C. Payoff Function

Let us now introduce the *reward-cost process* $\{Y_n\} = f^n(X_0, \ldots, X_n)$ where $f^n : \underbrace{\{0,1\} \times \ldots \times \{0,1\}}_{n-times} \rightarrow \mathcal{R}$ is an $\mathcal{F}_n$-measurable function. In the following we construct the processes of interest.

The random variable defined as $M_n := X_1 \cdot \ldots \cdot X_n,\ M_0 = 1$ is non-negative and can take values from the state space $\{0, 1\}$. If at some point $k,\ X_k = 0$ (ACK), then $M_{n \geq k} = 0$ that means that the process may only stay constant or decrease. Then the process forms a *super-martingale* [13]

$$\begin{aligned} \mathbf{E}[M_n | \mathcal{F}_{n-1}] &= X_1 \cdot \ldots \cdot X_{n-1} \cdot \mathbf{E}[X_n | \mathcal{F}_{n-1}] \\ &\leq X_1 \cdot \ldots \cdot X_{n-1} \\ &= M_{n-1} \end{aligned} \tag{8}$$

$\{-M_n\}$ forms a non-positive *sub-martingale*.

Suppose now that before each observation X_n we place a bet C_n the value of which we choose considering only the known observations $\{X_1(\omega) = x_1,\ X_2(\omega) = x_2,\ \ldots, X_{n-1}(\omega) = x_{n-1}\}$. Then since $\{C_n\}$ is $\mathcal{F}_{n-1}$ measurable and independent of X_n it forms a *previsible* process. Some interesting choices of the previsible process for the ARQ analysis could be some sequence of rates e.g. $C_n = R_o$, $C_n = \frac{R_o}{n}$ or $C_n = \beta^n R_o,\ 0 < \beta \leq 1$.

The reward that we receive for observing the random variable X_n equals $C_n \cdot (M_{n-1} - M_n)$. Then if $M_{n-1} = 1$ meaning that an ACK is not yet received up to step $n-1$ the n-th step reward can either equal C_n if $X_n = 0$ (ACK at step n) or 0 if the n-th retransmission is again unsuccessful. If $M_{n-1} = 0$ then definitely $M_n = 0$ and the n-th step reward is 0. The total reward up to n equals

$$Y_n = \sum_{k=1}^{n} C_k \cdot (M_{k-1} - M_k) \tag{9}$$

$$= \sum_{k=1}^{n} C_k \cdot X_1 \cdot \ldots \cdot X_{k-1} \cdot (1 - X_k) \tag{10}$$

The n-th step reward is the difference $Y_n - Y_{n-1} = C_n \cdot (M_{n-1} - M_n)$. Observe that the way we created the reward function (9) implies that if no ACK is received until n and given $M_0 = 1$ we have a total reward equal to $Y_n = 0$. If an ACK is received for the first time at some step $k \leq n$, then $Y_n = C_k$ and remains constant $\forall n\acute{} \geq n$. We can directly conclude

Lemma 1 *The reward process Y_n is a sub-martingale under the condition that $\{C_n\}$ is a non-negative, bounded and previsible process.*

Proof: We have already shown in (8) that $-M_n$ is a sub-martingale. Then, conditioned that $\{C_n\}$ is a non-negative, bounded, previsible process (see theorem 10.7 in [13]) we can directly deduce that

$$\mathbf{E}[Y_n - Y_{n-1} | \mathcal{F}_{n-1}] = C_n \cdot \mathbf{E}[-M_n + M_{n-1} | \mathcal{F}_{n-1}] \geq 0$$

■

Optimal stopping problems usually include a cost per observation as well as a terminal cost. For the case of ARQ it is reasonable to consider as cost the delay added to the system or the lost power due to an unsuccessful transmission. The costs per observation sum up and are deterministic, that is they do not depend on the values of the observed process directly. Their sum depends only on the number of observations we are willing to make. Furthermore the terminal cost is related to some penalty in the case we stop observation before a specific goal is attained - that is in our case a penalty for stopping before an ACK is received. Such a penalty is reasonable since the unsuccessful packet will be dropped and this will affect the user's quality of service. In the following we give a general expression of the *reward-cost* process, which will be denoted as the *payoff* at step n

$$Y_n^C = \sum_{k=1}^{n} C_k \left(M_{k-1} - M_k\right) - \gamma \sum_{k=1}^{n} D_k - \delta\mu_n \cdot M_n \quad (11)$$

Here, D_k are the non-negative costs per retransmission, μ_k are the terminal costs for stopping at stage k and $\gamma,\ \delta$ are constants indicating the relative importance of the costs over the rewards. These last two constants can be ommited if we consider that they are contained within D_k and μ_k. Furthermore different weights of the costs and penalty can reflect a different quality of service. That is a high value of γ can correspond to a service with high delay sensitivity, e.g. VoIP, while a high value of δ corresponds to services sensitive to dropped packets.

We are generally looking for a stopping rule $\tau : \{\tau \le n\} \in \mathcal{F}_n,\ \forall n \le \infty$ that decides when it is 'worth' stopping the process of retransmissions. Stopping at time n brings a payoff equal to Y_n^C. We are looking for a rule to maximize the expected value of this payoff.

III. The ARQ Stopping Problem without Cost

In this section we search for conditions to stop the countably infinite ARQ process with reward function $\{Y_n\}$ given by (10). This special case does not include any supplementary cost per retransmission or final cost. Since the previsible process is assumed bounded $C_n < K < \infty,\ \forall k$ by some $K > 0$ we have $\mathbf{E}\left[\sup_n Y_n^+\right] < \infty$ and the expectation of the reward function is well defined. Regarding the maximum expected reward, we can find that

Theorem 1 *For $\{C_k\}$ non-negative and upper-bounded by some $K > 0$ the maximum expected reward for the problem without cost equals*

$$V := \sup_\tau \mathbf{E}\left[Y_\tau\right] = \sum_{k=1}^{\infty} \left(C_k \cdot q_k \prod_{j=1}^{k-1} p_j \right) \le K \quad (12)$$

Proof: The stopped process is written as $\{Y_{\tau\wedge n}\}$ where $a \wedge b$ is the $\inf$ operator. We have seen in Lemma 1 that the process $\{Y_n\}$ under the conditions of theorem 1 is a submartingale. Furtermore from [13], every stopped submartingale is a submartingale. Since it is also bounded by K and $Y_{\tau\wedge n} \overset{n\to\infty}{\to} Y_\tau$ we can use the Bounded Convergence Theorem [13] to get $\mathbf{E}\left[Y_{\tau\wedge n}\right] \to \mathbf{E}\left[Y_\tau\right]$.

$$\begin{aligned} V &:= \sup_{\tau\in\mathcal{C}, 1\le\tau<\infty} \mathbf{E}\left[Y_\tau\right] \\ &= \sup_\tau \mathbf{E}\left[\sum_{k=1}^{\tau} C_k \cdot X_1 \dots X_{k-1}\left(1 - \mathbf{E}\left[X_k|\mathcal{F}_{k-1}\right]\right)\right] \\ &= \sum_{k=1}^{\infty} C_k \cdot q_k \left(\prod_{j=1}^{k-1} p_j\right) \le K \end{aligned}$$

■

From the expression of the maximum expected reward we observe that the value of the reward C_k achieved using the optimal stopping rule T is *geometrically distributed* with variable success probabilities. Hence, we can conclude that it is optimal to continue until the first ACK is received and immediately stop. This is intuitively clear due to the fact that $\{Y_n\}$ is a submartingale and remains constant and equal to C_k after the first ACK is received at time k. However for any Stopping Time, condition (1) has to be fulfilled. In our case this is equivalent to waiting only for finite steps a.s. until an ACK is received and can hold only when the chain in (7) is ergodic. Furthermore this can ensure that the stopping times remain a.s. finite in the ARQ problem with cost as well, to be investigated in the next section.

Theorem 2 *With probability 1 an ACK is fed back within a finite number of retransmissions, if the protocol described by the chain in (7) is ergodic.*

Proof: To prove this we initially require the expression of the mean recurrence time for state 1

$$\begin{aligned} E\left[K\right] &= \sum_{k=1}^{\infty} k \cdot f_{1,1}^{(k)} = \sum_{k=1}^{\infty} k \left(\prod_{l=1}^{k-1} p_l\right) q_k \\ &= q_1 + 2p_1q_2 + 3p_1p_2q_3 + \dots \\ &= 1 + \sum_{k=1}^{\infty} p_1p_2\dots p_k \end{aligned} \quad (13)$$

If the chain is ergodic, $E\left[K\right] < \infty$. The probability of the event $\mathcal{A}_n$ that more than n retransmissions are required until an ACK is fed back equals $P\left\{\mathcal{A}_n\right\} = P\left\{k > n\right\} = p_1 \cdot \ldots \cdot p_n$. Then the expression in (13) is an infinite sum of the probabilities of the events $\mathcal{A}_n$. From the first Borel-Cantelli Lemma [14] since the series in (13) converges, only finitely many of the events $\mathcal{A}_n$ can occur. ■

Three conditions for ergodicity of the chain are provided

Theorem 3 *If $\limsup_{k\to\infty} p_k < 1$ the chain in (7) is* ***ergodic****.*

Proof: The proof is derived from Foster's condition for ergodicity [14]. The condition is reduced to the convergence of the series $\sum_{k=1}^{\infty}\prod_{j=1}^{k} p_j$. Due to limited space details are omitted.

■

Theorem 4 *The following two sufficient conditions hold:*

- *If* $\lim_{k\to\infty}\sup(1-q_k\cdot k)<0$ *then (7) is* ***ergodic***.
- *If* $\lim_{k\to\infty}\inf(1-q_k\cdot k)>0$ *then (7) is* ***non-ergodic***.

Proof: The above theorem comes directly from the Foster-Lyapunov stability criterion [15] which extends Pakes' [16] sufficient conditions for ergodicity and also from Kaplan's [17] sufficient conditions for non-ergodicty.

Suppose the Lyapunov function used is $V(x)=x:\mathcal{N}_+\to\mathcal{N}_+$ and $\mathcal{D}$ is a finite subset of $\mathcal{N}_+$. From [15] we have that a Markov Chain is ergodic (positive recurrent) if $\epsilon>0$ and b is a constant such that the drift function

$$\begin{aligned}\gamma_k &= E\left\{V\left(Z_{t+1}\right)|V\left(Z_t=k\right)\right\}-V\left(k\right)\\ &\overset{V(x)=x}{=} \sum_{j=1}^{\infty}\left(j-k\right)P_{k,j}\leq -\epsilon+b\cdot I_{\mathcal{D}}\end{aligned}$$

$I_{\mathcal{D}}$ is the indicator function $I_{k\in\mathcal{D}}=1$. In the case of chain (7) $\gamma_k=1-k\cdot q_k,\ k=1,2,\ldots$ and $1-k\cdot q_k<1$. Then we can choose $b=1$. If $k\notin C$ then $1-k\cdot q_k\leq-\epsilon$ for infinitely many k's and does not hold only for a finite subset $\mathcal{D}$. From the definition of $\lim\sup$ [18] the condition can be written as $\lim_{k\to\infty}\sup(1-q_k\cdot k)<0$.

Using now Kaplan's conditions, a Markov Chain is non-ergodic if for some integer $N\geq1$ and constants $B\geq0,\ c\in[0,1]$ the following two conditions hold

$$\sum_{j=1}^{\infty}\left(j-k\right)P_{k,j}>0 \qquad \forall k\geq N$$

$$z^k-\sum_{j=1}^{\infty}P_{k,j}z^j\geq -B\left(1-z\right) \qquad \forall k\geq N,\ \forall z\in[c,1]$$

The conditions are reduced to

$$\begin{aligned}&\left(1-k\right)q_k+p_k=1-k\cdot q_k>0 && \forall k\geq N\\ &z^k-q_k\cdot z-p_k\cdot z^{k+1}\geq -B\left(1-z\right) && \forall k\geq N,\ \forall z\in[c,1]\end{aligned}$$

The second inequality is satisfied for $c=1,\ B=0$. From the first condition it is required that infinitely many k's satisfy the inequality $1-k\cdot q_k>0$ and only a finite subset of $\mathcal{N}_+$ should be left out. Then from the $\lim\inf$ definition [18] this reduces to $\lim_{k\to\infty}\inf(1-q_k\cdot k)>0$. ■

Upper-bounding the asymptotic probability of the tail events $\mathcal{A}_n$ that an ACK is not received up to n-th retransmission, a further sufficient condition for ergodicity can be provided.

Theorem 5 *If* $P\{k>n\}=p_1\cdot\ldots\cdot p_n=\mathcal{O}\left(\frac{1}{n^\beta}\right)$, $\beta>1$ *then the chain in (7) is ergodic.*

Proof: If $p_1\cdot\ldots\cdot p_n=\mathcal{O}\left(\frac{1}{n^\beta}\right)$, there exist constants $C,N^*>0$ s.t. $p_1\cdot\ldots\cdot p_n\leq C\cdot\frac{1}{n^\beta}$ holds $\forall n\geq N^*$ [19]. Using the Cauchy criterion for series convergence [18] and assuming $\beta>1$, we can see that $\sum_{k=n}^{m}p_1\cdot\ldots\cdot p_k\leq\sum_{k=n}^{m}C\cdot\frac{1}{n^\beta}=\epsilon<\infty$, $\forall m,n\geq N^*(\epsilon)$. Then $\sum_{k=1}^{\infty}\prod_{j=1}^{k}p_j$ converges and from Foster's criterion (proof of Th.3) the chain is ergodic. ■

IV. The ARQ Stopping Problem with Cost

If the success probabilities satisfy the Theorems of the previous section, the waiting time up to first ACK remains a.s. finite and equals (13) in average. In many cases however, where data are transmitted in real time and are delay sensitive, as for example in VoIP or Video Streaming, the maximum acceptable delay until a message is correctly received cannot exceed a specific upper bound, else communications fail. Furthermore, the value given from (13) can be rather large. In such cases it is reasonable to consider truncated ARQ protocols with a maximum accepted number of retransmissions. Then on the one hand we obtain a reduced number in average and we are sure that the delay will definitely not exceed the truncation number. On the other hand a packet may be dropped in case it is not accepted after the maximum defined number of efforts, thus having a certain cost in the service quality.

We are looking in this section for the optimal truncation time that can maximize an expected cost-reward function that reflects the aforementioned tradeoff. The expected payoff for the problem with cost is of course upper bounded by (12). We will have to loose some of the generality of the function suggested in (11) since the calculations would otherwise be rather complicated and the results not so neat. We consider from now on a constant cost per retransmission $\gamma\cdot D_n=D,\ \forall n$ and a constant final cost $\delta\cdot\mu_n=\mu,\ \forall n$. The simplified reward-cost function is now

$$Y_n^C=\sum_{k=1}^{n}C_k\cdot\left(M_{k-1}-M_k\right)-n\cdot D-M_n\cdot\mu \qquad (14)$$

We will first prove that the condition in (3) is satisfied

Lemma 2 *If the sequence* $\{C_n\}$ *is non-negative and upper-bounded by some value* $K>0$ *and* $\mu,D\geq0$, $\mathbf{E}\left[\sup_n\left(Y_n^C\right)^+\right]<\infty$.

Proof: Since $\left(Y_n^C\right)^+\leq Y_n$ a.s. under the conditions of Lemma 2, it obvously holds $\sup_n\left(Y_n^C\right)^+\leq\sup_n Y_n^+$ $\Rightarrow\mathbf{E}\left[\sup_n\left(Y_n^C\right)^+\right]\leq\mathbf{E}\left[\sup_n Y_n\right]<\infty$. ■

The above Lemma together with the ergodicity conditions guarantee the existence of an optimal stopping rule achieving the maximum expected reward $V^C:=\sup_\tau\mathbf{E}\left[Y_\tau^C\right]\leq V:=\sup_\tau\mathbf{E}\left[Y_\tau\right]\leq K$. The inequality holds since $Y_n^C\leq Y_n$ a.s.

The solution to the optimal stopping problem with cost can be simplified compared to the general dynamic programming solution if we can show that the problem is *monotone* (see Lemma 4 in [10]). If the sequence of rewards $\{Y_1,Y_2,\ldots\}$ is such that for every $n\geq\tau$

$$\mathbf{E}\left(Y_{n+1}|\mathcal{F}_n\right)\leq Y_n\Rightarrow\mathbf{E}\left(Y_{n+2}|\mathcal{F}_{n+1}\right)\leq Y_{n+1} \qquad (15)$$

we say we are in the monotone case and the optimal stopping rule is the *one-stage look-ahead (myopic) rule*.

$$T_1:=\min\left\{n:Y_n\geq\mathbf{E}\left[Y_{n+1}|\mathcal{F}_n\right],\ n\in\mathcal{N}_+\right\} \qquad (16)$$

Lemma 3 *The ARQ problem with cost having the simplified reward-cost function $\left\{Y_n^C\right\}$ is a monotone stopping problem as defined in (15), under the condition*

$$(C_n+\mu)\,q_n \geq (C_{n+1}+\mu)\,q_{n+1} \tag{17}$$

Proof: We are given that for some n it holds $\mathbf{E}\left[Y_n^C|\mathcal{F}_{n-1}\right] \leq Y_{n-1}^C$. This is reduced to

$$\mathbf{E}\left[Y_n^C|\mathcal{F}_{n-1}\right] = Y_{n-1}^C + \mathbf{E}\left[Y_n^C - Y_{n-1}^C|\mathcal{F}_{n-1}\right] \leq Y_{n-1}^C \Leftrightarrow$$

$$-D + (C_n+\mu)\,X_1\cdot\ldots\cdot X_{n-1}\left(1-\mathbf{E}\left(X_n|\mathcal{F}_{n-1}\right)\right) \leq 0 \tag{18}$$

We want to show under which conditions the inequality holds also for $n+1$. We have $\mathbf{E}\left[Y_{n+1}^C|\mathcal{F}_n\right] =$

$$\begin{aligned}
&= Y_n^C - D + (C_{n+1}+\mu)\,X_1\cdot\ldots\cdot X_n\left(1-\mathbf{E}\left(X_{n+1}|\mathcal{F}_n\right)\right)\\
&\overset{(18)}{\leq} Y_n^C - (C_n+\mu)\,X_1\cdot\ldots\cdot X_{n-1}\left(1-\mathbf{E}\left(X_n|\mathcal{F}_{n-1}\right)\right)\\
&\qquad + (C_{n+1}+\mu)\,X_1\cdot\ldots\cdot X_n\left(1-\mathbf{E}\left(X_{n+1}|\mathcal{F}_n\right)\right)\\
&= Y_n^C - X_1\cdot\ldots\cdot X_{n-1}\cdot\left[(C_n-\mu)\left(1-\mathbf{E}\left(X_n|\mathcal{F}_{n-1}\right)\right)\right.\\
&\qquad \left. - (C_{n+1}+\mu)\,X_n\cdot\left(1-\mathbf{E}\left(X_{n+1}|\mathcal{F}_n\right)\right)\right]
\end{aligned}$$

Then for the one-stage look-ahead rule to hold we demand

$$\begin{aligned}
&X_1\cdot\ldots\cdot X_{n-1}\cdot\left[(C_n+\mu)\left(1-\mathbf{E}\left(X_n|\mathcal{F}_{n-1}\right)\right)\right.\\
&\quad \left.-(C_{n+1}+\mu)\,X_n\cdot\left(1-\mathbf{E}\left(X_{n+1}|\mathcal{F}_n\right)\right)\right] \geq 0
\end{aligned} \tag{19}$$

This last inequality holds true if

- i) For some $k \leq n$ we have $X_k = 0$ (ACK received)
- ii) For all $X_k = 1,\ k \leq n$ (no ACK received). Then $(C_n+\mu)\,q_n \geq (C_{n+1}+\mu)\,q_{n+1}$.

∎

Theorem 6 *The optimal stopping rule for the monotone (17) problem with cost is to continue retransmissions until the first ACK is received or the inequality $q_{n+1} \leq \frac{D}{C_{n+1}+\mu}$ is satisfied for the first time.*

$$T = \min\left\{n \in \mathcal{N}_+ : X_n = 0 \ or\ q_{n+1} \leq \frac{D}{C_{n+1}+\mu}\right\} \tag{20}$$

Proof: Using the optimality of the one-stage look-ahead rule under the condition (17), we have from (18)

$$\begin{aligned}
\mathbf{E}\left[Y_{n+1}^C|\mathcal{F}_n\right] &= Y_n^C -\\
-D + (C_{n+1}+\mu)\,X_1\cdot\ldots\cdot X_n&\left(1-\mathbf{E}\left(X_{n+1}|\mathcal{F}_n\right)\right)
\end{aligned} \tag{21}$$

Then stopping occurs if

$$(C_{n+1}+\mu)\,X_1\cdot\ldots\cdot X_n\left(1-\mathbf{E}\left(X_{n+1}|\mathcal{F}_n\right)\right) \leq D \tag{22}$$

This last inequality holds true if

- i) For some $k \leq n$ we have $X_k = 0$ (ACK received)
- ii) For all $X_k = 1,\ k \leq n$ (no ACK). The stopping condition reduces to $(C_{n+1}+\mu)\cdot q_{n+1} \leq D$.

∎

Theorem 7 *The maximum expected reward for the ARQ problem with cost equals*

$$\begin{aligned}
V^C &:= \sup_\tau \mathbf{E}\left(Y_\tau^C\right) = \mathbf{E}\left(Y_T^C\right) \qquad (23)\\
&= \sum_{k=1}^{n^*}\left[C_k + (n^*-k)\,D + \mu\right] q_k \prod_{j=1}^{k-1} p_j - n^*D - \mu
\end{aligned}$$

where

$$n^* = \max\left\{n \in \mathcal{N}_+ : q_n > \frac{D}{C_n+\mu}\right\} \tag{24}$$

Proof: Observe that the n^* comes from the optimal stopping rule in (20) and is the maximum $n \in \mathcal{N}_+$ for which retransmissions are allowed and afterwards we immediately stop. Then

$$\begin{aligned}
\mathbf{E}\left(Y_T^C\right) &= \sum_{k=1}^{n^*}\mathcal{P}\left(T=k\right)\cdot Y^C\left(T=k\right)\\
&= \sum_{k=1}^{n^*} q_k \prod_{j=1}^{k-1} p_j\left(C_k - kD\right)\\
&+ \left(-n^*D-\mu\right)\prod_{j=1}^{n^*} p_j
\end{aligned}$$

and after some calculations we reach (23). ∎

V. Some Special Cases of Stopping

In the current paragraph we use the main result for the optimal stopping time expressed in (20) to investigate some special Truncation Times.

1. *No retransmissions allowed - the Case $T = 1$*

This case occurs when the optimal stopping time $T = 1$, that is stopping occurs after the first transmission and no ARQ takes place. From the truncation rule in (20) this occurs when $q_2 \leq \frac{D}{C_2+\mu}$. The expected reward equals ($n^* = 1$)

$$V_1^C := (C_1+\mu)\cdot q_1 - (D+\mu) \tag{25}$$

2. *No communications allowed - the Case $T = 0$*

Observe that if the value of V_1^C in (25) above is negative not even the first transmission should take place since it is more probable that we have a cost rather than a positive gain. This happens when

$$0 > (C_1+\mu)\cdot q_1 - (D+\mu) \Rightarrow q_1 < \frac{D+\mu}{C_1+\mu} \tag{26}$$

For $D > C_1$ we get that $q_1 \leq 1$ hence regardless the success probability of the first transmission no communications should take place.

3. *Infinite Truncation Time*

The case where no truncation should take place occurs when the inequality $q_n > \frac{D}{C_n+\mu}$ holds true $\forall n \in \mathcal{N}_+$. For the case where $q_n = q$:=const. we can see that $C_n \geq 0 \Rightarrow \frac{D}{\mu} \geq \frac{D}{C_n+\mu}$. Hence if $q > \frac{D}{\mu}$ then no finite truncation time exists.

VI. Applications

In the following we apply the optimal stopping criterion given in (20) to specific reward sequences $\{C_n\}$ and success probabilities $\{q_n\}$ that correspond to practical ARQ scenarios. The section will provide a better understanding of the impact of choice for the costs D and μ.

A. Constant success probabilities $q_n = q$:=const and effective coding rates $C_n = \frac{R}{n}$

In the case of ARQ retransmissions a message is sent at each time slot/trial with a rate R bits/sec/Hz. Suppose that the message is correctly received after n retransmissions and $X_n = 0$ (ACK), $X_{m<n} = 1$ (NAKs). We say that the effective coding rate for the current message equals $\frac{R}{n}$ [2], [5]. The rewards for the process equal in this case $C_n = \frac{R}{n}$. The probability of success per retransmission is considered constant. The random variable denoting the number of retransmissions until correct reception is geometrically distributed.

We have to show first the optimality of the one-stage look-ahead rule. The criterion in (17) holds true for the specific choice of C_n and q_n of interest

$$\left(\frac{R}{n} + \mu\right) \cdot q \geq \left(\frac{R}{n+1} + \mu\right) \cdot q \tag{27}$$

The solution of (20) is rewritten as

$$n^* = \left\lceil \frac{R}{\frac{D}{q} - \mu} \right\rceil - 1 \tag{28}$$

where $\lceil \ldots \rceil$ is the ceiling function, under the condition that $\frac{D}{q} - \mu > 0$. Observe that if $q \geq \frac{D}{\mu}$ we fall into the case of infinite truncation times of Section V.3. Furthermore the stopping rule can be $T > 0$ - in other words transmissions are allowed - if as shown in (26)

$$q \geq \frac{D + \mu}{R + \mu} \tag{29}$$

The above two conditions on D and μ, given R and q provide the range of D given μ

$$\mu \cdot q < D \leq R \cdot q - \mu \cdot p \tag{30}$$

Concerning the range of values of μ

$$0 \leq \mu \leq R \cdot q \tag{31}$$

The solution in (28) implies that given μ and D costs that lie within the accepted range (30) and (31) the optimal number of retransmissions increases when the supported rate and/or the success probability increases. On the other hand given a desired rate to support as well as a success probability the optimal number of retransmissions decreases as the cost per trial D increases or the terminal cost μ decreases. This is reasonable since a higher cost per trial discourages a high truncation number while a higher dropping cost urges for more attempts until the message is eventually correctly received.

The above remarks can be illustrated in the following two figures. In the first one Fig.2 $\mu = 0.5$ is kept fixed while the optimal number of retransmissions is shown to vary with R and D. In the second one Fig.3 the same scenario holds for $\mu = 0$, where the packet dropping is cost free. In both cases $q = 0.9$.

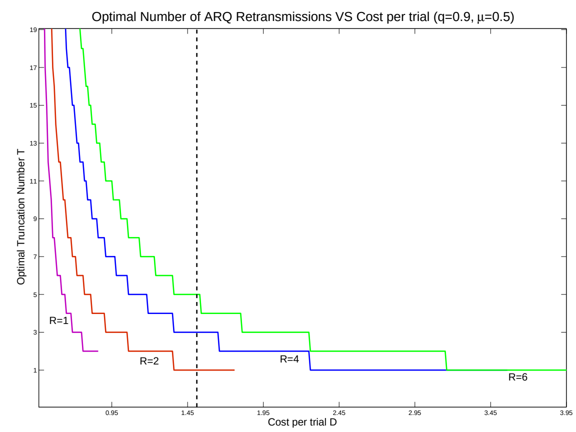

Fig. 2. Optimal Stopping time for different supported rates $R = \{1\ 2\ 4\ 6\}$, $\mu = 0.5$, $q = 0.9$ and $C_n = \frac{R}{n}$ VS Cost per trial D. Given that $D = 1.5$ we get that $T_{R=2} = 1$, $T_{R=4} = 3$, $T_{R=6} = 5$

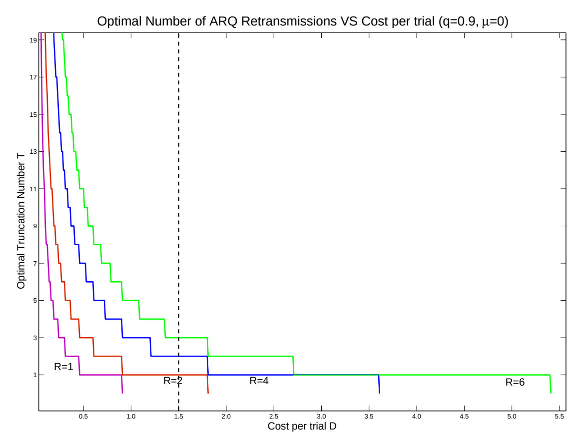

Fig. 3. Optimal Stopping time for different supported rates $R = \{1\ 2\ 4\ 6\}$, $\mu = 0$, $q = 0.9$ and $C_n = \frac{R}{n}$ VS Cost per trial D. Given that $D = 1.5$ we get reduced stopping times compared to $\mu = 0.5 > 0$ since no cost for dropping is inserted. $T_{R=2} = 1$, $T_{R=4} = 2$, $T_{R=6} = 3$

B. Exponentially decreasing error probabilities $p_n = e^{-\beta n}$ and discounted rate gain $C_n = R \cdot e^{-\alpha n}$, $\alpha, \beta > 0$

The error probability given that finite length codewords are transmitted can be shown to be upper-bounded by an exponential function $p \leq e^{-\beta \cdot N}$. The error exponent $\beta > 0$ shows how fast the error vanishes as the length of the code N tends to infinity. It is shown in [20] that in the case of incremental redundancy ARQ the error exponent of the upper bound, keeping the code length fixed and finite N :=const$< \infty$ increases proportional to the number of retransmissions, namely equals $k \cdot \beta$ if we currently are at retransmission effort $k \geq 1$. Then $p_k \leq e^{-\beta \cdot k}$, where we merge $N \cdot \beta := \beta$. Generally β depends on the supported rate R. In the following we assume that the error probability equals this upper bound. Furthermore we consider a sequence of discounted gains $R \cdot e^{-\alpha \cdot n}$ where the gain R which is the transmission rate per trial, is stepwise discounted by a factor $e^{-1} < 1$. In what follows $\mu = 0$.

The $\alpha, \beta > 0$ should satisfy

$$\beta + \alpha < \alpha \cdot e^{\beta} \tag{32}$$

so that the 1-stage look-ahead rule to be optimal. Then stopping occurs for the minimum n that satisfies the following inequality

$$1 \leq \frac{D}{R} e^{\alpha n} + e^{-\beta n} \tag{33}$$

The range of D for $\mu = 0$ equals

$$0 = \mu \leq D \leq R \cdot e^{-a} \left(1 - e^{-\beta}\right) \tag{34}$$

In the following keeping $\alpha, \beta > 0$ fixed so that they satisfy (32) (specifically $\alpha = 0.8, \beta = 0.5$) we plot the optimal stopping times for different supported rates as a function of cost D. The cost varies within (34).

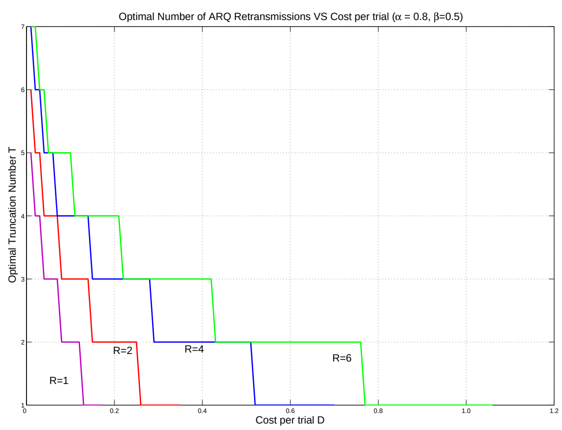

Fig. 4. Optimal Stopping time for different supported rates $R = \{1\ 2\ 4\ 6\}$, $\mu = 0$, $\alpha = 0.8, \beta = 0.5$ and $C_n = R \cdot e^{-\alpha n}$, $q_n = 1 - e^{-\beta n}$ VS Cost per trial D. Given that $D = 0.2$ we get $T_{R=2} = 2$, $T_{R=4} = 3$, $T_{R=6} = 4$

VII. Conclusions

We have described an ARQ protocol with different success probabilities per trial as a discrete-time Markov Chain with countably infinite state space and have further formulated the problem of truncating the chain of retransmissions as an optimal stopping problem. A reward-cost process has been constructed as a function of the sequentially observed ACK/NAK feedback. The rewards can be related to some rate gain whereas the costs to delay/power consumption per trial (D_k) as well as to dropped packets (μ_k) in case the ARQ process is stopped before correct packet reception. Parameters D and μ depend on the quality of service to be supported. Solution of the ARQ stopping problem without cost showed that continuing retransmissions up to first ACK maximizes the expected reward. Several conditions for the ergodicity of the ARQ chain in (7) were provided. These guarantee finite retransmission efforts until correct packet reception with probability 1. Truncated ARQ protocols keep the delay definitely below a specific allowable retransmission number at a cost of a positive packet dropping probability. Solution of the ARQ problem with cost provided an explicit expression of the optimal truncation time as a function of the costs and rewards. Cases where no retransmissions as well as where infinite truncation times should be allowed have been investigated. The optimal stopping rules have been applied to two ARQ scenarios. For the first one, constant success probabilities and effictive coding rates equal to $\frac{R}{n}$ were considered. In the second one, an incremental redundancy ARQ protocol with exponentially decreasing error probabilities and discounted coding rates was investigated. The optimal retransmission number was shown to have an increasing behavior with respect to rate, probability of success and dropping cost and a decreasing behavior as the cost per trial increases.

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