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Stochastic Geometry: modeling and analysis of wireless networks

Anastasios Giovanidis

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Contact: anastasios.giovanidis@lip6.fr

Bibliography

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How to work with Interference?

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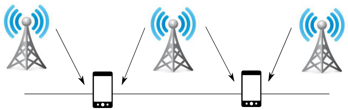
- In wireless networks Interference is a source of quality degradation.
- Other users, devices, stations, WIFI, etc. transmit on the same frequency band.
- How to take this into account in our models/analysis?



Wyner Model

Grid and Hexagonal Cell

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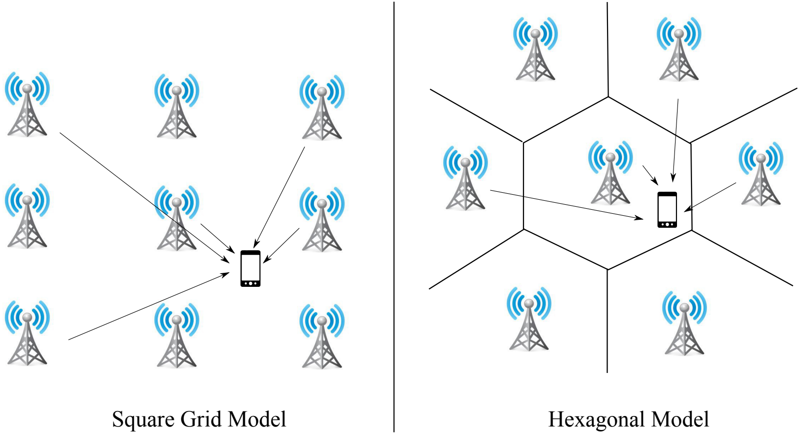

First order interference

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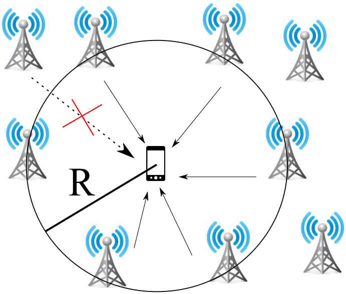
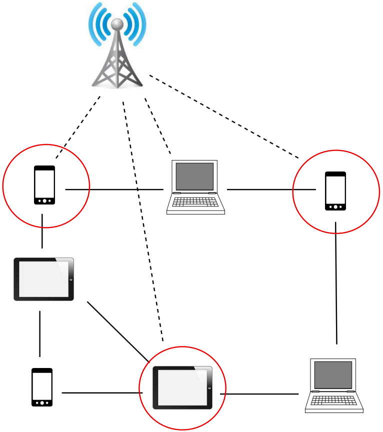


First order interference
(equidistant)



Interference Graph

Empirical

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- Summarize Interference influence in a single random variable (r.v.)

$$
\begin{aligned}
SINR &= \frac{S_o}{N + I_o} \\
S_o &= P H_o R_o^{-\beta} \\
I_o &= \sum_n I_{n,o}.
\end{aligned}
$$

where $I_o \sim \mathcal{X}$, an empirical r.v. (from fitting).

- Alternatively exhaustive simulations (user/station positions, fading, noise, mobility, frequency reuse etc.) e.g. NS3 simulator for LTE.

Point Processes

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- Characterise interference by probabilistic modelling of the node positions.
- Randomness of node position in space (2D, 3D, ++D)
- Results describe the whole distribution, not just an instance.
- Performance (outage, throughput, delay)

$$\mathbb{P}(\mathrm{SINR} > T), \ \mathbb{P}(R > \rho), \ \mathbb{P}(D > \delta).$$

- Not only averages.
- Not only given topologies. Any topology drawn from a given distribution.

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Point Process in 1D (a)

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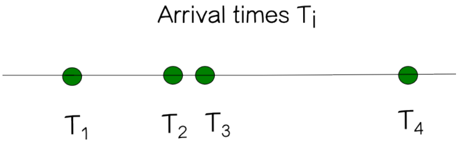
- One dimension (e.g. "time"). Sequence of random times when a particular event occurs.
- Example "emergency calls in a hospital".
- Given a period of time there will be a random number of such calls.



Point Process in 1D (b)

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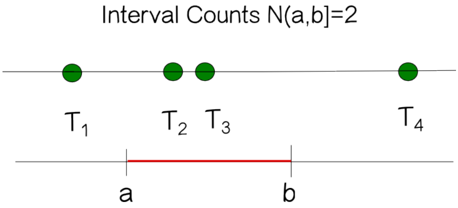
- 1D Point Processes (PP) has a natural ordering $T_1 < T_2 < \ldots$.
- Inter-arrival times: $S_i = T_{i+1} - T_i$ (independent for some)
- Cumulative counting process: $N_t = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \leq t\}}$

N_t: total number of points up to time t, for all $t \geq 0$



Point Process in $\geq$ 2D

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- In higher dimensions there is no natural ordering of points.
- No natural analogue for inter-arrival times S_i, counting process N_t.
- BUT! Generalise the interval counts to region counts.

Point Process in $\geq$ 2D

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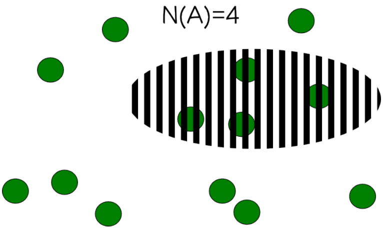
- In higher dimensions there is no natural ordering of points.
- No natural analogue for inter-arrival times S_i, counting process N_t.
- BUT! Generalise the interval counts to region counts.

$$N(A) = \textit{number of points falling in } A, \quad A \subset \mathbb{R}^d$$



PP Fundamentals (a)

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- Φ: A random, finite or countably-infinite collection of points in $\mathbb{R}^d$.
- Realization: Discrete subset $\phi = \{x_i\} \subset \mathbb{R}^d$.
- Counting measure: $N(A)$ gives the number of points of ϕ in A. $\mathbf{1}_{\{x \in A\}}$ is the Dirac measure at x.

$$N(A) = \sum_i \mathbf{1}_{\{x_i \in A\}}$$

PP Fundamentals (b)

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- Simple PP: $\mathbb{P}\left[N(\{x\}) \leq 1,\ \forall x\right] = 1$, i.e. the points $\{x_i\}$ are pairwise different a.s. No two points coincide.

- The distribution of a PP Φ is entirely characterised by the family of finite distributions $\{N(A_1), \ldots, N(A_k)\}$, with $A_1, \ldots, A_k$ bounded.

PP Characteristics

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- The intensity (mean) measure

$$\Lambda(A) \quad = \quad \mathbb{E}[N(A)].$$

- The empty (void) probability

$$\mathcal{V}(A) \quad = \quad \mathbb{P}[N(A)=0], \; A \in \mathcal{B}.$$

Example PP 2D

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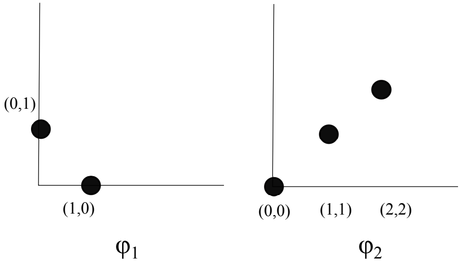
- Φ can have two realisations: ϕ_1 ($p = 1/4$), ϕ_2 ($p = 3/4$).
- A_1 is the ball $\mathcal{B}((1,1), 1.5)$. A_2 is $\mathcal{B}((0,0), 0.5)$.
- $N(A_1) = 2$ ($p = 1/4$) and $N(A_1) = 3$ ($p = 3/4$).
- $\Lambda(A_1) = 2 \cdot (1/4) + 3 \cdot (3/4) = 11/4$.
- $\mathcal{V}(A_2) = 1 \cdot (1/4) + 0 \cdot (3/4) = 1/4$.



Probability Generating Functional (PGFL)

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- Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$.

$$\mathcal{P}_\Phi(f) = \mathbb{E}\left[\prod_{X_i \in \Phi} f(X_i)\right] \qquad (PGFL)$$

- Ex. $f(x) = ||x - (0,0)||^2$ for a given point $x \in \mathbb{R}^2$.

$$\mathcal{P}_\Phi(||x - (0,0)||^2) = (1/4) \cdot 1 \cdot 1 + (3/4) \cdot 0 \cdot 2 \cdot 8 = 1/4.$$

Laplace Transform

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- PGFL can be used to compute the Laplace Transform (LT) of a function F,

$$F = \sum_{X_i \in \Phi} g(X_i).$$

$$\mathcal{L}_f(s) = \mathbb{E}[e^{-sF}] = \mathbb{E}\left[\exp(-s \sum_{X_i \in \Phi} g(X_i))\right] = \mathbb{E}\left[\prod_{X_i \in \Phi} \exp(-sg(X_i))\right].$$

PGFL Application

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- PGFL used to calculate the LT of the Interference in a wireless network.
- Downlink. Φ are the stations. Let a user at the origin $(0,0)$.

$$\begin{aligned} g(X_i) &= \frac{PH_i}{||X_i - (0,0)||^\beta}. \\ F = I_o &= \sum_{X_i \in \Phi} \frac{PH_i}{||X_i - (0,0)||^\beta}. \end{aligned}$$

Campbell's Formula

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- Let $f : \mathbb{R}^d \to \mathbb{R}$.

$$\mathbb{E}\left[\sum_{X_i \in \Phi} f(X_i)\right] = \int_{\mathbb{R}^d} f(x)\,\Lambda(dx)\,.$$

$$\stackrel{(int.\ fun.)}{=} \int_{\mathbb{R}^d} f(x)\,\beta(x)\,dx.$$

If the intensity measure Λ satisfies $\Lambda(A) = \int_A \beta(x)\,dx$ for some function β, then the latter is the intensity function of Φ.

$$\mathbb{P}[N(dx) > 0] \approx \mathbb{E}[N(dx)] \approx \beta(x)\,dx.$$

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Definition

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- For every compact set $A \subset \mathbb{R}^d$ the count $N(A)$ has a Poisson distribution with mean $\Lambda(A) = \lambda \mathcal{S}(A)$.
- If $\Lambda(dx) = \lambda dx \Rightarrow$ homogeneous. $\Lambda(A) = \lambda \mathcal{S}(A)$.
- Complete Independence Property
  If $A_1, \ldots, A_k$ are mutually disjoint compact sets, then $N(A_1), \ldots, N(A_k)$ are independent random variables.

Example: Homogeneous Poisson, 2D, $\lambda = 1$, $A = \mathcal{B}((0,0), R = 1)$, $\mathcal{S}(A) = \pi$.

$$\mathbb{P}(N(A) = n) = e^{-\lambda\pi} \frac{(\lambda\pi)^n}{n!}$$

Examples

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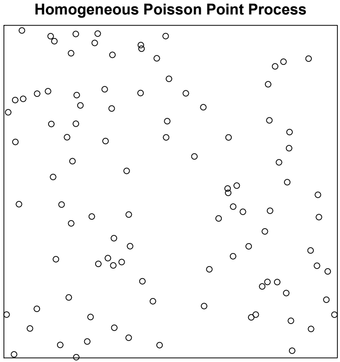

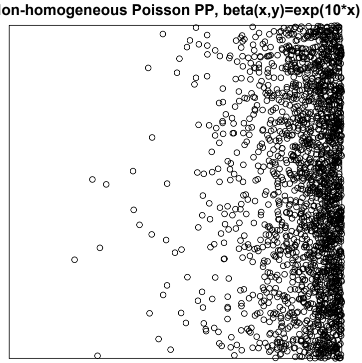

Simulating the Poisson PP

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How to simulate a Poisson PP of intensity λ in window W?

1. First generate a random variable with Poisson distribution and mean $\lambda \mathcal{S}(W)$.

2. Given the realisation of the Poisson r.v. n, generate n independent uniform random point in W.

Campbell's mean formula

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- Example: Homogeneous Poisson PP with $\beta(x) = \lambda$.
- Mean Interference from disc $A = \{x : a \leq ||x|| < b\}$ seen at the origin $(0, 0)$. $H = 1$ const.

$$\begin{aligned}
\mathbb{E}[I] = \mathbb{E}\left[\sum_{X_i \in \Phi \cap A} f(X_i)\right] &= \int_A f(x)\,\lambda dx \\
&= \int_A P||x||^{-\beta}\lambda dx \\
&= 2\pi\lambda P \int_a^b r^{-\beta+1} dr. \\
&= \frac{2\pi\lambda P}{2-\beta}(b^{2-\beta} - a^{2-\beta}).
\end{aligned}$$

- If $b \to \infty$ then interference finite iff $\beta > 2$.

PGFL

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- Let $f : \mathbb{R}^d \to \mathbb{R}$.

$$\mathcal{P}_{\Phi}(f) = \mathbb{E}\left[\prod_{X_i \in \Phi} f(X_i)\right] = \exp\left(-\lambda \int_{\mathbb{R}^d} (1 - f(x))dx\right).$$

- Laplace Transform

$$\mathbb{E}[e^{-sF}] = \mathcal{P}_{\Phi}(e^{-sg}) = \exp\left(-\lambda \int_{\mathbb{R}^2} (1 - e^{-sg(x)})dx\right).$$

Example: Poisson PP, Interference, $\beta = 4$.

$$\mathbb{E}[e^{-sI}] = \exp\left(-2\pi\lambda \int_0^{\infty} (1 - e^{-sPr^{-4}})rdr\right) = \exp(-\pi\lambda\sqrt{\pi sP}).$$

Poisson law preservation

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- Superposition: $\Phi_S = \Phi_1 + \Phi_2$. Then,

$$N_S(A) = N_1(A) + N_2(A), \qquad \Lambda_S(A) = \Lambda_1(A) + \Lambda_2(A).$$

- Independent Thinning: $\mathbb{P}[\Delta(X_i) = 1] = \delta > 0$.

$$N_T(A) = \sum_{X_i \in \Phi} \Delta(X_i) \mathbf{1}_{\{X_i \in A\}}, \qquad \Lambda_T(A) = \delta \Lambda(A).$$

- Mapping/Displacement: If each point of a PPP is randomly perturbed with the same law then the resulting process is also PPP. (e.g. $X_i = X_i + \Sigma_i$, Σ_i is a 2D random Gaussian vector).
- Clustering: If each point of a PPP is replaced by a random finite set of points Z_i, then the resulting process is also PPP.

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Palm Measure

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- Interested in properties relating to a typical point of the process.
- Conditional probabilities of events **given there is a point of the PP at a specific location**.

The **Palm probability $\mathbf{P}^x(\Gamma)$** of an event Γ at location x is the conditional probability that the event Γ will occur, given $x \in \Phi$.

$$\mathbf{P}^x(\Gamma) \quad = \quad \mathbb{P}^x(\Phi \in \Gamma)\,.$$

Let us remove the extra point x from the consideration. The reduced Palm distribution $\mathbf{P}^{!x}$ of a PP Φ is the distribution of $\Phi \setminus x$ under $\mathbf{P}^x$:

$$\mathbf{P}^{!x}(\Gamma) \quad = \quad \mathbf{P}^x(\Phi \setminus \{x\} \in \Gamma)\,.$$

Slivnyak-Mecke Theorem

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- The Poisson PP, under the Palm distribution $\mathbf{P}^x$, behaves as if it were a Poisson PP superimposed with a fixed point at the location x

$$\mathbf{P}^x = \mathbf{P} * \Delta_x \quad (\textit{convolution})$$
$$\Phi^x \stackrel{d.}{=} \Phi \cup \{x\}.$$

- Φ is a Poisson PP **if and only if**

$$\mathbf{P}^{!x} = \mathbf{P}.$$

- In the context of a cellular downlink network, it allows to treat the interference as coming from the entire Poisson PP, despite removing the serving BS from the interfering set.

Nearest Neighbour (a)

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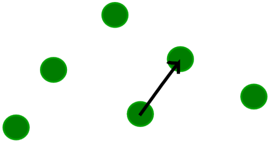
Q: What is the probability distribution of
- the distance from some atom of Φ to its nearest neighbour atom?



Nearest Neighbour Distance

Nearest Neighbour (b)

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- Formally $R_x = dist\,(x, \Phi \setminus x)$, $x \in \Phi$. Find:

$$\mathbb{P}\left(R_x \leq r | x \in \Phi\right).$$

- Non-elementary conditioning: The event $\{x \in \Phi\}$ has 0 probability.

Nearest Neighbour (Intuition)

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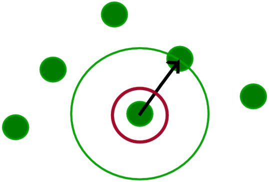
- Let Φ be Poisson PP with intensity λ.
- Take a small neighbourhood around 0, e.g. the ball $\mathcal{B}(0,\epsilon)$.
- $R_\epsilon = dist\left(0, \Phi \setminus \mathcal{B}\right)$
  the distance from 0 to the nearest point of Φ outside $\mathcal{B}$.

$$\begin{aligned} \mathbb{P}\left[R_\epsilon > r | N\left(\mathcal{B}\left(0,\epsilon\right)\right) > 0\right] &= \exp\left\{-\lambda\pi\left(r^2 - \epsilon^2\right)\right\} \\ \mathbb{P}\left[R_0 > r | 0 \in \Phi\right] &= \exp\left\{-\lambda\pi r^2\right\}. \end{aligned}$$



Nearest Neighbour Distance

Nearest Neighbour (Palm)

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For a stationary PP Φ in $\mathbb{R}^d$, the nearest neighbour function G is the distribution of the distance from a typical point $x \in \Phi$ to the nearest other point of Φ

$$G(r) \quad = \quad \mathbf{P}^x\left(dist\left(x, \Phi \setminus \{x\}\right) \leq r\right) = \mathbf{P}^x\left(N(\mathcal{B}\left(x, r\right) \setminus \{x\}) > 0\right).$$

- If Poisson PP we may use *Slivnyak's* theorem $\mathbf{P}^{!x} = \mathbf{P}$

$$\begin{aligned} G(r) &= \mathbb{P}^0\left(dist\left(0, \Phi \setminus \{0\}\right) \leq r\right) = \mathbf{P}\left(dist\left(0, \Phi\right) \leq r\right). \\ &= 1 - \exp\left(-\lambda \pi r^2\right), \end{aligned}$$

equal to the empty space function.

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Marked PP

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- The points of a PP might be labelled with extra information called marks.
- e.g. If emergency calls are a PP on the line, each point might carry a label with the place of the call and the nature of emergency.
- Marked point is a pair: $\{x_i, m_i\}$, $m_i \in M \subseteq \mathbb{R}^k$.
- The marked PP is denoted by $\tilde{\Phi}$ and has counting measure $\tilde{N}(\cdot)$, i.e.

$$\tilde{N}(A \times M) = \sum_i \mathbf{1}_{\{x_i \in A, m_i \in M\}}.$$

- Thinning a PP is formalised by a marked point process with marks in $\{0, 1\}$. $\{0\}$=retain and $\{1\}$=delete.

Marked PP Example

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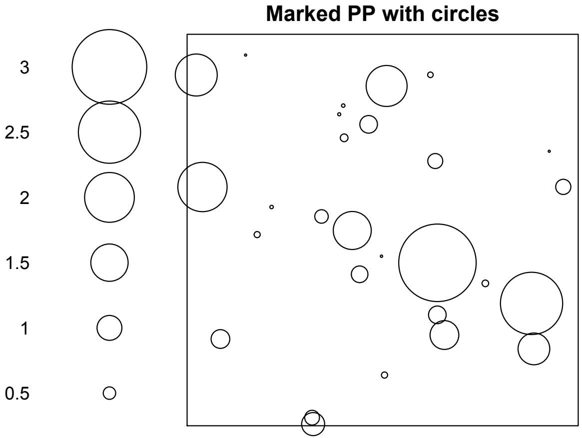

Independent Marking

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- Independently Marked PP (i.m.PP): If given the locations of the points $\Phi = \{x_i\}$, the marks are mutually independent random vectors in M and if the conditional distribution satisfies

$$\mathbf{P}\left(m \in \cdot | \Phi\right) = \mathbf{P}\left(m \in \cdot | x\right) = Q_x\left(dm\right).$$

- The Laplace functional of an i.m. Poisson PP for all functions in $\mathbb{R}^+$ is

$$\begin{aligned} \mathcal{L}_{\tilde{\Phi}}\left(f\right) &= \mathbb{E}\left[e^{-\sum_i f(x_i, m_i)}\right] \\ &= e^{-\int_{\mathbb{R}^d}\left(1-\int_M e^{-f(x,m)} Q_x(dm)\right)\Lambda(dx)}. \end{aligned}$$

Marked Stationarity

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- For a stationary marked PP, the intensity measure takes the form

$$\mathbb{E}\left[\tilde{N}\left(A \times M\right)\right] \quad = \quad \lambda \mathcal{S}\left(A\right) Q_0\left(M\right).$$

$\lambda =$ expected number of points per unit area, $Q_0(M) =$ distribution of the typical mark (probability measure).

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END